Radiometric and signal-to-noise ratio properties of multiplex dispersive spectrometry

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Recent theoretical investigations have shown important radiometric disadvantages of interferential multiplexing in Fourier transform spectrometry that apparently can be applied even to coded aperture spectrometers. We have reexamined the methods of noninterferential multiplexing in order to assess their signal-to-noise ratio (SNR) performance, relying on a theoretical modeling of the multiplexed signals. We are able to show that quite similar SNR and radiometric disadvantages affect multiplex dispersive spectrometry. The effect of noise on spectral estimations is discussed. © 2010 Optical Society of America
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1. Introduction
Multiplexing spectrometry [1,2] was developed in the early 1950s as a remedy for the lack of array detectors in the infrared spectral interval, where the measurement required a time-consuming scanning process to collect all the spectral wavebands of interest. Hence, spectrometry was much more inefficient in the infrared (IR) than in the visible spectral range, where the availability of photographic films permitted instead the simultaneous acquisition of all spectral samples. The adaptation of a telecommunication technique made it possible to multiplex all the spectral samples through the unique photosensitive element, obtaining a higher amplitude signal and a proportional reduction of the integration time. The need for exactly invertible multiplexing drove the adoption of a set of orthogonal functions, which would provide the multiplexed signal of the observed source with the requested characteristics.

Multiplexing received two alternative implementations: two-beam interferometers and coded aperture dispersive spectrometers. Two-beam interferometers [Fourier transform spectrometry (FTS)] implement multiplexing by means of the set of harmonic functions, producing the cosine transform of the spectrum of the observed source. In contrast, aperture coded dispersive spectrometers can implement various sets of orthogonal functions, such as the Walsh (Hadamard), harmonic, and Legendre polynomials [3,4]. Usually, the spectrum of the observed source modulates the selected set of orthogonal functions that are coded as a bidimensional spatial pattern of transmittance (or reflectance) in the input and/or the output aperture.

The main difference between FTS and multiplex dispersive (MD) spectrometers is that FTS devices realize interferometric amplitude multiplexing [4], while dispersive spectrometers put into operation intensity multiplexing. Moreover, spectral integration (multiplexing) is performed by Fourier transform (FT) spectrometers, even when they are equipped with both input and output slits, as different from dispersive instruments, where multiplexing specifically requires the absence of the input or the output slit. As a common point, any multiplex spectrometer measures not the spectrum itself, but a complex transformation of it. Hence, such instruments require specific data preprocessing for transforming back the observed data into the desired spectral

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estimations (i.e., the spectrum of the observed source). A number of research programs and scientific initiatives have focused on the opportunity to adopt multiplex spectrometers in the past 60 years, and nowadays FTS is probably the widest field in spectrometry. MD spectrometry (MDS) has also gained broad popularity, and Hadamard multiplexing (also known as HTS or Hadamard transform spectrometry) has even been applied to mass spectrometry [5], Raman imagery [6], and image and movie coding.

We describe a novel theoretical analysis of MDS in order to investigate in more depth the radiometric and signal-to-noise ratio (SNR) properties of this spectrometric technique. We demonstrate that the important radiometric limitations of FTS pointed out by recent research [7] can be almost entirely applied also to MDS, significantly lightening the claimed radiometric advantages of MDS in terms of signal amplitude (étendue) and SNR. Performance comparison between MDS and traditional spectrometry shows that intensity multiplexing does not bring any real advantage to the measurement process, giving rise instead to important disadvantages.

2. Mathematical Methods and Naming Convention

MD spectrometers produce an integral transform \( I(x) \) of the incoming specific intensity \( i(\lambda) \), as shown in the following relationship:

\[
I(x) = \begin{cases} 
\int_0^{+\infty} i(\lambda) F(x, \lambda) d\lambda + N(x) & \forall x \in D_x \\
0 & \text{elsewhere}
\end{cases},
\]

(1)

where \( F(x, \lambda) \) is the integral kernel of the implemented (forward) transform, \( N(x) \) is the noise due to the detector and readout circuitry, and \( x \) is the focal plane coordinate (conjugate of the wavelength \( \lambda \) that is measured within a limited interval \( D_x \). The source spectral estimation (back transformation) is then obtained by performing the inverse integral transform, by means of the analysis kernel \( B(x, \lambda) \) [4]:

\[
i(\lambda) = \int_0^{+\infty} I(x) B(x, \lambda) dx \quad \forall \lambda \in D_\lambda,
\]

(2)

where the symbol \( D_\lambda \) designates the observed spectral interval. We adopt a lowercase symbol to represent a signal in the spectral domain (\( \lambda \)), and the corresponding uppercase symbol for its transform in the conjugated spatial domain (\( x \)). Symbols \( F, f \) are reserved for the forward transform kernel; \( B, b \) always describe the inverse transformation. In view of Eqs. (1) and (2) we have

\[
\int_0^{+\infty} F(x, \lambda) B(x, \nu) dx = \delta(\lambda - \nu) \quad \forall \lambda \in D_\lambda,
\]

\[
\int_0^{+\infty} F(x, \lambda) B(\xi, \lambda) d\lambda = \delta(x - \xi) \quad \forall x \in D_x,
\]

(3)

a feature of these integral kernels that has also been the focus of earlier studies [3,8]. Let us note that we embed in this modeling only real valued functions and constants, while integrals are extended to the real positive semi-axis. This last condition is rather arbitrary, so all integrals in the following analysis can be thought to span the entire real axis. The previous equations should be intended as a mathematical framework of multiplex spectrometers that avoids the complexity arising from a detailed analysis of the optical characteristics of specific implementations of MDS. As an example, the above modeling does not care about the port onto which multiplexing is physically implemented. The specific aim of our work is to analyze in depth the properties of the multiplexed signal using a mathematical representation that can be applied to nearly any MD spectrometer.

Let us note that some authors employ similar equations to describe the basic properties of various MD spectrometers [3,8,9], while many others adopt a discrete modeling for taking into account signal transformations [4,10–12]. As an example, Eq. (6) in the work of McCain et al. [8] is nearly the same as our Eq. (1).

As already noted, every implementation of MDS adopts a complete orthogonal set of functions \( e(x, \lambda) \) for a practical definition of the two integral kernels \( F(x, \lambda) \) and \( B(x, \lambda) \) that must obey Eq. (3). The selected orthogonal functions are always normalized \((-1 \leq e(x, \lambda) \leq 1)\) and comply with the following properties:

\[
\langle e(x, \lambda) \rangle_x = \int_0^{+\infty} e(x, \lambda) dx = \delta(\lambda),
\]

\[
\langle e(x, \lambda) \rangle_x = \int_0^{+\infty} e(x, \lambda) d\lambda = \delta(x),
\]

(4)

where we have also defined a compact notation \( \langle \cdot \rangle \) to indicate spectral and spatial integrals. General orthogonality relationships among the functions \( e(x, \lambda) \) are not reported in Eq. (4); i.e., \( \langle e(x, \lambda) e(x, \nu) \rangle_x = \delta(\lambda - \nu) \). The following equations give a general enough definition that can describe almost every MD spectrometer:

\[
F(x, \lambda) = F_0 e(x, \lambda) + f(x, \lambda),
\]

\[
B(x, \lambda) = B_0 e(x, \lambda) + b(x, \lambda),
\]

(5)

where \( F_0 \) and \( B_0 \) are real constants. There are two main reasons for using such a complex definition of the two integral kernels with two additional functions \( f(x, \lambda) \) and \( b(x, \lambda) \). First, most of the orthogonal functions \( e(x, \lambda) \) that can be used for MDS assume negative values that cannot be optically coded as a transparency or reflectance mask [3], whose values are limited to the range \([0, 1]\). Therefore, the function \( f(x, \lambda) \) and the constant \( F_0 \) are chosen so that the
forward integral kernel follows the constraints below:

\[ 0 \leq F(x, \lambda) \leq 1 \quad \forall \ \lambda \in D_x, \quad \forall \ x \in D_x. \]  

(6)

The supplementary function \( b(x, \lambda) \) and the constant factor \( B_0 \) guarantee that the transformation is invertible, i.e., it obeys Eq. (3). An obvious consequence of this circumstance is that

\[ F_0B_0 = 1. \]  

(7)

A similar reasoning has been developed by Gehm et al. [3], who clearly show that \( F(x, \lambda) \) and \( B(x, \lambda) \) cannot be the same set of auto-adjoint orthogonal functions. Second, defining the multiplexing without the function \( f(x, \lambda) \) would not add any radiometric advantage to the multiplexed signal \( I(x) \), as long as every integral transform based only on a complete set of orthogonal functions \( e(x, \lambda) \) must obey Plancherel’s theorem. It is worth noting that, in an MD spectrometer, the measured quantity is the source transform (datagram) \( I(x) \), and possible radiometric advantages must be related to the observation of a signal of higher power than \( i(\lambda) \). In fact, a criterion for choosing optimal (luminosity) forward transform kernels is that its spectral average should be greater than or equal to 0.5 [9], \( \langle F(x, \lambda) \rangle_\lambda = \langle f(x, \lambda) \rangle_\lambda \geq 0.5 \), or maximal [8,12].

Many implementations [3,8,12] of MDS use the optical configuration detailed in the following equations:

\[ F_0 = \frac{1}{2}, \quad B_0 = 2, \quad f(x, \lambda) = \frac{1}{2}, \quad b(x, \lambda) = 0. \]  

(8)

Let us note that the same values in Eqs. (8) can even represent signals in a FT spectrometer. In this case, the orthogonal functions adopted are the set of harmonic functions.

The proof of Plancherel’s theorem for a nonspecific integral transform \( e(x, \lambda) \) and a generic signal \( s(\lambda) \leftrightarrow S(x) \) is straightforwardly deduced by the orthogonal property of functions \( e(x, \lambda) \) themselves:

\[ \int_0^{+\infty} \int \! s(\lambda) \bigl| f(\lambda, \nu) |B_0 e(\lambda, \lambda + b(\lambda, \lambda)) \bigr| \, dx \, d\lambda 
= \int_0^{+\infty} [s(\lambda)] S(x) e(\lambda, x) \, dx \, d\lambda 
\]  

(9)

Let us note that, as long as the measurement range \( D_x \) has a finite extension, only a limited subset of the orthogonal functions \( e(x, \lambda) \) is implied in the measurement of the datagram \( I(x) \). In this way, the actual set of orthogonal functions is incomplete and the datagram \( I(x) \) would obey Bessel’s inequality \( \langle F^2(x) \rangle_\lambda \leq \langle I^2(\lambda) \rangle_\lambda \), a circumstance that has been already noted previously [3]. Presumably, this condition is largely because the instrument does not measure the components at the higher spatial frequencies that compose the spectrum \( i(\lambda) \); hence, the loss of signal power is truly related to a loss in spectral resolution. The finite resolving power is an unavoidable feature of any spectrometer, and the corresponding loss of signal power is a secondary issue for many spectroscopic applications. For the sake of simplicity and considering its minor relevance, we will neglect this characteristic in the remainder of our analysis, assuming that Plancherel’s theorem is always obeyed. As pointed out by Gehm et al. [3], this theorem is exactly obeyed for discrete code masks, such as those employed in HTS. When considering a different set of orthogonal functions, the equations that will be deduced under this approximation will exactly hold true if the input spectrum \( i(\lambda) \) source is defined as a low-pass prefiltered copy (smoothing) of the observed source.

3. Characteristics of Signals in a MD Spectrometer

In this section we study the principal properties of the multiplexed signal using the mathematical framework depicted in Section 2. More specifically, we are going to show remarkable disadvantages connected with the use of the multiplexing scheme.

A. Radiometric Effects of Multiplexing

We stem our analysis by substituting Eqs. (5) into the first of Eqs. (3), obtaining

\[ \int \! [F_0 \delta(x, \nu) + F(x, \nu)] B_0 e(\lambda, \lambda) + b(\lambda, \lambda) \, dx \]  

(10)

where \( \lambda \) is the estimation wavelength and \( \nu \) is a generic wavelength of the arriving intensity that is multiplexed to obtain the datagram \( I(x) \). Equation (10) admits a reliable solution if the function \( f(x, \lambda) \) is orthogonal to the analysis function \( B(x, \lambda) \), and \( b(x, \lambda) \) is orthogonal to \( F(x, \lambda) \) within the interpolated spectral domain \( D_\lambda \) at least. In other words, the three terms originated by the functions \( f(x, \lambda) \) and \( b(x, \lambda) \) would not add any contribution to the spectral estimates, and the product \( F_0B_0 \) must equal 1 as stated in Eq. (7):

\[ B_0 \delta(x, \nu) b(x, \lambda) \lambda = 0 \quad \forall \ \lambda \in D_\lambda, \]  

\[ F_0 \delta(x, \nu) b(x, \lambda) \lambda = 0 \quad \forall \ \lambda \in D_\lambda, \]  

(11)

\[ \{f(x, \nu) b(x, \lambda) \lambda = 0 \quad \forall \ \lambda \in D_\lambda. \]  

The most common condition encountered in MDS has been detailed in Eqs. (8), with \( f(x, \lambda) = f_0 = 1/2 \) and
\(b(x, \lambda) = 0 \ [8-10,12]\), and produces an extended equation set that holds true for any \(\lambda\):

\[
\int_0^\infty |F_0 e(x, \nu) + f(x, \nu)|B_0 e(x, \lambda) + b(x, \lambda)|dx
= F_0 B_0 \delta(\lambda - \nu) + B_0 f_0 \delta(\lambda),
\]

\[
F_0 \langle e(x, \lambda) i(\lambda) \rangle_x = \langle f(x, \nu) b(x, \lambda) \rangle_x = 0,
\]

(12)

where we have also utilized the property of functions \(e(x, \lambda)\) reported in Eqs. (4). Moreover, since the term \(f(x, \lambda)\) is set to a constant value \(f_0 = 1/2\), the datagram \(I(x)\) can be written as

\[
I(x) = F_0 \langle e(x, \lambda) i(\lambda) \rangle_x + f_0 \langle i(\lambda) \rangle_x.
\]

(13)

The first term on the right-hand side of Eq. (13) yields the integral transform originated by the orthogonal set of functions \(e(x, \lambda)\) only, and the last term, \(f_0 \langle i(\lambda) \rangle_x\), is a constant value proportional to the panchromatic source luminosity. As shown in Eqs. (11) and (12), this last term does not give contributions to the spectral estimates because its inverse transform is a Dirac delta pulse lying outside the interpolated spectral interval \(D_s\). But it is even more remarkable that the same term does not contain information pertaining to the shape of the source spectrum.

In other words, the multiplexed signal is made up of two components, \(I(x) = S(x) + U(x)\). The first, \(S(x)\), is the integral transform term \(S(x) = F_0 \langle e(x, \lambda) i(\lambda) \rangle_x\) associated with the selected orthogonal functions \(e(x, \lambda)\) (Walsh functions, frequently). The signal \(S(x)\) holds the spectral information about the observed source, and its inverse transform is the spectrum estimator, so we call it the informative signal. The second component is the constant term \(U(x) = f_0 \langle i(\lambda) \rangle_x\). The signal \(U(x)\) does not contain any information regarding the source, apart from its luminosity, and its inverse transform term is zero in the interpolation interval \(D_s\). We define this contribution as the useless or noninformative part of the measured multiplexed signal \(I(x)\). It is worth noting that the useless term \(U(x) = f_0 \langle i(\lambda) \rangle_x\) carries the main contribution to the measured datagram in terms of signal power. This characteristic of the multiplexing architecture is easily deduced, considering that MDS has to keep a significant radiometric advantage over nonmultiplexing spectrometers at the level of physical signal. On the other hand, Plancherel’s theorem constrains the informative component \(S(x) = F_0 \langle e(x, \lambda) i(\lambda) \rangle_x\) to have approximately the same energy as the spectrum of the observed source \(i(\lambda)\) \((F_0^2 \sim 1/4)\), which is just the signal available for a nonmultiplexing spectrometer. The two conditions combine together in the following manner.

The multiplex advantage for the physical signal can be stated as

\[
\langle I^2(x) \rangle_x \gg \langle i^2(\lambda) \rangle_x, \quad \langle I^2(x) \rangle_x = \langle S^2(x) \rangle_x + \langle U^2(x) \rangle_x.
\]

(14a)

Due to Plancherel’s theorem, we can write

\[
\langle S^2(x) \rangle_x = F_0^2 \int_{D_s} |\langle e(x, \lambda) i(\lambda) \rangle_x|^2 dx = F_0^2 \langle i^2(\lambda) \rangle_x,
\]

(14b)

thus we obtain the following outcome:

\[
\langle I^2(x) \rangle_x \cong \langle U^2(x) \rangle_x \Rightarrow \langle S^2(x) \rangle_x, \quad \int_{D_s} |f_0 \langle i(\lambda) \rangle_x|^2 dx = F_0^2 \int_{D_s} |\langle e(x, \lambda) i(\lambda) \rangle_x|^2 dx.
\]

(14c)

The MD spectrometer makes the energy of the datagram signal \(\langle I^2(x) \rangle_x\) high above that of its informative component \(S(x) = F_0 \langle e(x, \lambda) i(\lambda) \rangle_x\), which, in turn, has almost the same amplitude as the input spectrum \(\langle i^2(\lambda) \rangle_x\) that could be measured by a nonmultiplexing instrument. Therefore, most of the datagram energy is carried by the noninformative signal component, as stated by Eq. (14c). Additional information regarding the signal amplitude can be achieved by introducing the spectral dispersion \(\gamma^2_{(i(\lambda))}\) of the source \(i(\lambda)\); i.e., \(\gamma^2_{(i(\lambda))}\) is a measure of the source spectral variability, not a statistical standard deviation. With simple mathematical steps it results in

\[
\gamma^2_{(i(\lambda))} = \frac{\langle i^2(\lambda) \rangle_x}{\mu(D_s)} - \left[ \frac{\langle i(\lambda) \rangle_x}{\mu(D_s)} \right]^2 \Rightarrow \langle i(\lambda) \rangle_x \mu(D_s)
= \mu(D_s) \langle i^2(\lambda) \rangle_x - \mu^2(D_s) \gamma^2_{(i(\lambda))}.
\]

(15)

Here we have introduced the measure \(\mu(D_s) = \int_D dx\) of the interpolation interval (spectral range). Substituting Eq. (15) into the first of Eqs. (14c), and indicating the measure of the measurement interval with \(\mu(D_s) = \int_{D_s} dx\), we have

\[
\int_{D_s} I^2(x) dx \cong \int_{D_s} f_0^2 \langle i(\lambda) \rangle_x^2 dx = f_0^2 \mu(D_s) \langle i(\lambda) \rangle_x^2
= f_0^2 \mu(D_s) |\langle i^2(\lambda) \rangle_x - \mu^2(D_s) \gamma^2_{(i(\lambda))}|.
\]

(16)

and solving for \(\langle i^2(\lambda) \rangle_x\) yields the following:

\[
\langle S^2(x) \rangle_x = F_0^2 \langle i^2(\lambda) \rangle_x
\]
\[
\int_{D_s} I^2(x) dx + f_0^2 \mu(D_s) \mu^2(D_s) \gamma^2_{(i(\lambda))} \cong \int_{D_s} I^2(x) dx + f_0^2 \mu(D_s) \mu^2(D_s) \gamma^2_{(i(\lambda))}.
\]

(17)

We point out that the energy of the informative component of the datagram signal [the left-hand side in Eq. (17)] has the same order of magnitude as the datagram energy (the energy of the overall multiplexed signal) divided by the product \(\mu(D_s) \mu(D_s)\). The product \(\mu(D_s) \mu(D_s)\) is proportional to the number of
samples in the datagram by the number of samples in the spectrum, two numbers that are often equal \(4, 11, 12\). Roughly speaking, the amplitude term \(1/\sqrt{\mu(D_\eta)\mu(D_\eta)}\) can be compared to the amplitude radiometric gain introduced by the open input aperture of the dispersive spectrometer. Apparently, the informative signal \(F_0\langle e(x, \lambda)|\langle \lambda \rangle\rangle\), entirely loses the radiometric advantage that MDS has over traditional nonmultiplexing instruments at the level of the physical signal.

Finally, we point out that Barducci et al. [7] reached similar conclusions with regard to the radiometric and noise characteristics of FT spectrometers. As already noted, the modeling developed for the MDS in the current work is straightforwardly applicable even to FTS, so the conclusions and the signal properties drawn in this section are easily generalized to nearly any kind of multiplex spectrometer.

B. Noise in MDS

In an MD spectrometer, noise is originated in the datagram (spatial) domain, but we are further interested in its effects in the conjugated spectral domain. The passage from the datagram to the source spectrum is performed using the analysis (interpolation) function \(B(x, \lambda)\) that, as stated in Eqs. (5) and (8), is usually the orthogonal function \(B_0 e(x, \lambda)\). Considering that \(B_0\) is a constant factor that only affects the amplitude of the interpolated spectrum, the inverse transform is, therefore, unitary and, as such, obeys many standard properties of unitary integral transforms obtained by an orthogonal function set.

The first property of interest is that additive white noise in the measurement domain always transforms into a white noise field (process) that degrades the spectral estimations. A proof of this property has been described by Papoulis [13] for the case of the FT and is easily extended to a generic integral transform \(e(x, \lambda)\).

We show an example of this demonstration that is very short. Let \(N(x)\) be the measurement white noise with autocorrelation \(H_{NN}(x, \xi) = N_0\delta(x - \xi)\) and \(n(\lambda)\) its inverse transform by means of the orthogonal functions \(e(x, \lambda)\). Using the symbol \(E\{\}\) to indicate the statistical ensemble average operator, we can compute the autocorrelation \(H_{nn}(\lambda, \nu)\) of the field \(n(\lambda)\) as

\[
H_{nn}(\lambda, \nu) = E\left\{ \int_0^{+\infty} N(x)e(x, \lambda)dx \int_0^{+\infty} N(\xi)e(\xi, \nu)d\xi \right\}
\]

\[
= \int_0^{+\infty} \int_0^{+\infty} E\{N(x)N(\xi)e(x, \lambda)e(\xi, \nu)d\xi dx
\]

\[
= \int_0^{+\infty} N_0\delta(x - \xi)e(x, \lambda)e(\xi, \nu)d\xi dx
\]

\[
= N_0\delta(\lambda - \nu).
\]

This equation reveals that \(n(\lambda)\) again is a white noise field having the same power (variance) as the measurement noise \(N(x)\).

The second point of great interest concerns how the measurement SNR transforms to the spectral domain. It has been shown [3] that the average SNR is invariant for FT, using a demonstration that only requires the applicability of Plancherel’s theorem. Having shown that this theorem holds true also for a nonspecific integral transform \(e(x, \lambda)\), we are able to generalize the property of SNR invariance to the case of MDS. We write

\[
\text{SNR}_{\text{datagram}}^2 = \frac{\int_0^{+\infty} S^2(x)dx}{\int_0^{+\infty} N^2(x)dx} = \frac{\int_0^{+\infty} S^2(\lambda)d\lambda}{\int_0^{+\infty} N^2(\lambda)d\lambda} = \text{SNR}_{\text{spectrum}}^2.
\]

Therefore, the analysis of the random component of the spectroscopic signal can be performed equivalently in one of the two domains (datagram or spectrum), and the results are simply extended to the other one. Let us note that the SNR of the physical signal is much higher than the SNR of the informative signal in MDS. This point is addressed in the next subsection.

1. Photonic Noise

The presence of the high-power noninformative signal is critical when the dominant noise source is photonic, as in the visible and near-IR spectral ranges. This point has been addressed thoroughly in a work of Barducci et al. [7] about FTS multiplexing, and the results shown in that paper can be straightforwardly applied also to MDS. Here we give a brief description of the modeling discussed in that work, with some adaptation necessary for representing MD instruments.

The first circumstance to be considered is that the constant term \(U(x) = f_0\langle i(\lambda)\rangle\) does not add information to the source spectrum estimation, but the corresponding photonic noise cannot be separated from the informative component of the signal. Since the noninformative term holds most of the power of the measured signal, it absorbs most of the dynamic range of the employed detector while its photonic noise might be large with respect to the tiny informative component of the signal.

The photon flux \(\Phi\) experienced by a MD spectrometer is related to the datagram intensity and obeys Poisson’s statistics, giving rise to a flux variability whose standard deviation is just the square root of the flux itself. The standard deviation \(d\sigma_{\Phi(x)}\) of the monochromatic photon flux in the narrow spectral interval \(d\lambda\) impinging in a unitary time interval over a sensor having unitary equivalent area and field of view can be written as

\[
d\sigma_{\Phi(x)} = \sqrt{\frac{\lambda i(\lambda)}{c h}} \left\{ F_0 e(x, \lambda) + f_0 \right\} d\lambda,
\]

where \(c\) is the speed of light and \(h\) is Planck’s constant. When the above variability is multiplied by
the photon energy and the rms is summed over all the spectral elements \(d\lambda\), the overall photonic noise affecting the datagram \(d\sigma f\) is obtained:

\[
\sigma_f = \sqrt{\int_0^{+\infty} \frac{ch}{\lambda} \left\{ F_0 e(x, \lambda) + f_0 \right\} d\lambda.}
\] (21)

Equation (21) allows us to write the maximum effective MDS SNR, \(\text{SNR}_{\text{eff max}}(x)\), allowed by the photonic noise only:

\[
\text{SNR}_{\text{eff max}}(x) = \frac{F_0 \left\langle i(\lambda) e(x, \lambda) \right\rangle_{\lambda}}{\sqrt{\int_0^{+\infty} \frac{ch}{\lambda} \left\{ F_0 e(x, \lambda) + f_0 \right\} d\lambda.}}
\] (22)

Let us note that the signal amplitude in the Eq. (22) (numerator) has been taken equal to the informative component of the signal, so we get an effective SNR, whereas the SNR characteristic of the physical signal is much higher. Vice versa, the photonic noise term in the denominator has been estimated, neglecting the small contribution introduced by the informative signal \(S(x)\). The obtained result puts an upper limit to the effective SNR achieved by MDS, since it only holds random contributions from photonic noise. Therefore, the actual SNR of the informative signal of a MD spectrometer has to be less than the value of \(\text{SNR}_{\text{eff max}}(x)\). Calculating the average of \(\text{SNR}_{\text{eff max}}^2(x)\) originates the following equation:

\[
\text{SNR}_{\text{eff max}}^2 = \frac{\left\langle \text{SNR}_{\text{eff max}}^2(x) \right\rangle_{\lambda}}{\mu(D_x)} \cong \frac{F_0^2 \left\langle \left\langle i(\lambda) e(x, \lambda) \right\rangle^2 \right\rangle_{\lambda}}{chf_0 \mu(D_x) \left\langle \frac{\left\langle i(\lambda) \right\rangle_{\lambda}^2}{\lambda} \right\rangle_{\lambda}}
\] (23)

where we used Plancherel’s theorem to estimate the numerator. To facilitate the comparison with a nonmultiplexing device, we report a rough approximation to the average \(\text{SNR}_{\text{nonmult max}}^2\) computed as the ratio of the average square signal to the average square noise of photonic origin:

\[
\text{SNR}_{\text{nonmult max}}^2 = \frac{\left\langle \left\langle i^2(\lambda) \right\rangle_{\lambda} / \mu(D_x) \right\rangle}{ch \left\langle \frac{\left\langle i(\lambda) \right\rangle_{\lambda}}{\lambda} \right\rangle / \mu(D_x)}.
\] (24)

Equation (24) shows that the nonmultiplexing device has an advantage in terms of average SNR as large as a \(\sqrt{\mu(D_x)}\) factor, which is proportional to the number of collected spectral channels. Thus, the performance of MDS shows an important limit when the photonic noise is examined. The advantage of nonmultiplexing dispersive techniques is due to the circumstance that it only inputs random contributions from its effective signal. Therefore, the utilization of MDS in the visible spectral range, where the photonic noise has a larger effect, does not appear to be a viable strategy for high spectral resolution measurements. Nevertheless, in the IR spectral range, the photonic noise is mitigated by the lower photon energy, thus high spectral resolution multiplex observations may still be a possible alternative to nonmultiplexing instruments.

4. Discussion and Concluding Remarks

The theory of MDS has been reexamined with the specific aim to analyze possible radiometric and SNR advantages of this technique with respect to traditional nonmultiplexing spectrometry. This research has been inspired by a previous work [7] regarding possible similar advantages of the FTS. The two issues are really strictly linked because, usually, FTS and MDS claim Fellgett’s and Jaquinot’s advantages among their motivations [3,11,14–16], both being multiplex spectrometric techniques.

We have developed a theoretical model of signals characteristics of a MD spectrometer that is general enough to represent almost every optical configuration of MDS, and even signals in FTS. In this model, we have been able to identify the presence of a noninformative signal that brings most of the datagram energy. We have shown also that the residual informative signal carries a power that roughly equals the power available for a nonmultiplexing dispersive spectrometer. This last circumstance implies the loss of any radiometric advantages induced by the multiplex technology. The same behavior has been unveiled previously in FTS [7]. Moreover, we have shown that the noninformative datagram (or interferogram) component originates enhanced photonic noise, which makes the effective SNR of multiplex spectrometers significantly worse than that attained by nonmultiplexing instruments, in the limit for a negligible detector noise. We point out that previous investigations concerning possible radiometric and SNR advantages of HTS and FTS found untrue results frequently, because they failed to recognize the occurrence of a large amplitude noninformative part in the multiplexed signal [17].

An additional common disadvantage of FTS and MDS (including HTS) is that the spectrometer has to measure a tiny signal (as a nonmultiplexing spectrometer does), which, however, is superimposed on a high-amplitude continuous radiation plateau. From a practical standpoint, this feature represents a weakness, because most of the digitalization accuracy of the employed detector is utilized to sense this useless high signal. Efficient MDS should adopt finer quantization accuracy than traditional nonmultiplexing spectrometry.
Frequently, the radiometric and SNR advantages of MDS and FTS are stated as a consequence of the reduction of the necessary integration time induced by the higher level of physical signal available in multiplex spectrometers [4]. In its turn, the reduction of integration time implies an advantage in SNR. Considering the outcomes of our theoretical modeling, the premise of this reasoning appears to be untrue. We remark that the detector’s integration time and quantization accuracy should be selected by considering the radiometric resolution necessary for gathering the requested information from the sole informative component \( S(x) \) of the multiplexed signal. If the integration time and the quantum amplitude (digital resolution) are regulated by considering the entire physical signal, the measured informative component of it might have poor radiometric accuracy. In this perspective, multiplexing does not imply any reduction of the integration time necessary to grab the desired spectral and radiometric information about the source, since the informative signal amplitude is truly slightly lower than that achieved by a nonmultiplexing spectrometer. In its turn, this behavior means that, when correctly operated, a multiplexing instrument will receive the same detector noise amplitude as a nonmultiplexing device, giving rise to a similar effective SNR in all those experimental conditions in which photonic noise \( n_{\text{phot}} \) is far below the detector noise level \( n_{\text{det}} \). We recap our results concerning the SNR comparison in the following equation:

\[
\begin{align*}
\text{SNR}_{\text{multiplexing}} & = \frac{\text{SNR}_{\text{nonmult}}}{\sqrt{\mu(O_i)}} \quad n_{\text{phot}} \gg n_{\text{det}}, \\
\text{SNR}_{\text{multiplexing}} & = \frac{\text{SNR}_{\text{nonmult}}}{n_{\text{phot}} \ll n_{\text{det}}},
\end{align*}
\]

(25)

This means that multiplexing always is disadvantageous in terms of SNR. The entire above schematic of signals in MDS is nearly the same obtained for FTS [7], permitting a large generalization of these theoretical results to any known multiplexing techniques. We notice that, according to a recent investigation [18], the radiometric advantage of multiplexing at the level of physical signal is also limited by the constant radiance theorem, at least for imaging (multiple modes) instruments.

An additional wrong assumption sometimes found in papers about MDS is that the counts (digital numbers) measured by the instrument when observing the multiplexed signal should be preserved during the computer operated estimation procedure [9] (meaning that the estimated spectrum should be output with such a high-amplitude level). This assumption is erroneous, because most of these counts are originated by the noninformative signal component, and should not be considered when estimating the spectrum of the observed source.

MD spectrometers have been implemented also by adopting the set of harmonic functions [3], originating FT multiplexing, as in two-beam spectrometers. In this case, the theory of multiplexing is exactly the same governing the FTS, and the multiplexed signal is thus subject to supplementary limitations. The explored range of the spatial coordinate \( x \) (or optical path difference in FTS) is strictly related to the obtained spectral resolution, and finer spectral details are made available after extending the measurement to a larger \( x \). Unfortunately, the cosinelike transformation operated by the harmonic MD spectrometer concentrates the power of the informative component of the signal at lower \( x \), making much more difficult the measurement of the datagram (interferogram) at large optical path differences that demand higher radiometric resolution. This behavior typical of the FT has been proved by Barducci et al. in lemma 1 of their paper [7], and the disadvantage originated by it has not been discussed in the previous sections since it only concerns a particular multiplexing implementation. In view of this property, the efficiency of harmonic MDS and FTS is very low in comparison with that attained by traditional dispersive spectrometers when high spectral resolution measurements are performed.

Finally, it is worth noting that, while nonmultiplexing spectrometers seem to provide the best radiometric and SNR performance, the above lemma 1 suggests that FTS would be inferior to nonharmonic MDS when the radiometric performance is considered.

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References