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A TRIBUTE TO

GIULIANO TORALDO DI FRANCIA

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To the memory of Giuliano Toraldo di Francia

Preface

This special issue is intended as a tribute to Giuliano Toraldo di Francia, who died recently. For many of us, he was a great teacher who taught us not only physics but also ethics, honesty, and scientific exactness. All those who knew him had the opportunity to appreciate both his great scientific mind and also his human depth.

We shall miss him.

The work presented here is a reproduction of a work by Toraldo di Francia. Despite the fact that it was written in 1957, it still appears very up-to-date. Even if the subject treated was published in due time, this specific work is unknown to the general reader because it was published as an internal report of the "Centre for the Study of Microwaves" (which then became "Institute of Research on Electromagnetic Waves" (IROE) and is now known as "Nello Carrara" Institute of Applied Physics" (IFAC)) of the Italian National Research Council.

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G.TORALDO DI FRANCIA

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S U M M A R Y

The spin of electromagnetic radiation can be revealed at microwave frequencies by the torque exerted by a circularly polarized wave upon a screen which can conduct current only parallel to a given direction. In order to make an accurate quantitative measurement, one has to know the angular momentum cross section of the screen. The first few terms of a series expansion of the cross section are evaluated, with the assumption that the screen is a circular disc whose radius is small with respect to the wavelength. In order to assess the accuracy with which the polarization must be circular, also the imaginary part of the cross section is evaluated. Explicit expressions are given for the scattered field over the screen. The feasibility of the experiment is discussed.

ON A MACROSCOPIC MEASUREMENT OF THE SPIN OF ELECTROMAGNETIC RADIATION

Introduction.

It was shown by Beth [1] that the spin of electromagnetic radiation can be revealed by a macroscopic measurement. He succeeded in measuring the torque exerted by a suitably polarized beam of light on a doubly refracting plate and found a good agreement with the theory.

A different experiment was suggested by Carrara [2], who made use of microwaves. He revealed the torque that a circularly polarized wave exerts upon a plane screen (or a system of plane screens) which can conduct the current only parallel to a given direction.

As is well known, field theory predicts that a circularly polarized wave should carry a spin per unit time per unit surface given by S/ω , where S is the magnitude of Poynting's vector and ω the angular frequency. If the wave carries N photons per unit time per unit surface, we shall have $S = Nh\omega/2\pi$, whence the spin carried by each photon turns out equal to $h/2\pi$, as it should be.

Apart from microwave applications (power measurements) it is interesting to make an accurate measurement of the angular momentum carried by the wave, since this represents an indirect measurement of the spin of the photons.

Now in Carrara's experiment the torque is substantial and can be easily revealed. However, an accurate agreement with the theory has not yet been found, simply because an accurate theory is not yet available. The size of the screen is necessarily of the same order of magnitude as the wavelength. Accordingly, one cannot make use of the geometrical cross section, but has to know the effective cross section for angular momentum absorption. This was pointed out by the author [3], who evaluated the principal part of the angular momentum cross section of a circular disc, whose radius is very small compared to the wave-

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(1) R.A.BETH, Phys. Rev., 50, 115 (1936)

(2) N.CARRARA, Nuovo Cimento, 6, 50 (1949); Nature, 164, 882 (1949)

(3) G.TORALDO DI FRANCIA, Nuovo Cimento, 3, 1276 (1956)

length. But, in order to be able to carry out an accurate measurement, one ought to make sure that the first term approximation is valid for the disc employed. This can only be accomplished by evaluating some of the following terms of the series expansion of the cross section.

It was also shown ([4]) that in order to assess the accuracy with which the polarization must be circular, one has to know the <u>imaginary</u> cross section as well.

It is the purpose of the present paper to evaluate some terms beyond the first for both the real and the imaginary cross sections. This will enable us to discuss the feasibility of an accurate experiment with a screen of small size.

<u>Recurrent Integro-Differential Equation for the Series Expansion of the Current.</u>

The screen Σ will be assumed to be an infinitely thin disc of radius r, which has infinite conductivity parallel to a given direction and is perfectly insulating in the perpendicular direction.

We shall refer to a rectangular system of coordinates x,y,z with x in the direction of conductivity and z coincident with the axis of the disc. The unit vectors in the directions of x, y, z will be denoted by $\underline{i}$, $\underline{j}$, $\underline{k}$ respectively.

Let the incident wave have elliptic polarization and travel in a direction $\underline{s} = \underline{j} \sin \varphi + \underline{k} \cos \varphi$, which makes an angle φ with z and is parallel to the plane yz. The wave may be split into two linearly polarized waves, one with the electric field parallel to x, and one with the electric field perpendicular to x. The second wave will not be scattered by Σ and, for the time being, we can dispense with its consideration. The first wave will be assumed to have the form

$$(1) \qquad \underline{E}^{i}(P) = \underline{i} \exp\left[ik\, \underline{s}\cdot(P-O)\right] = \underline{i} \exp\left[ik(y \sin \varphi + z \cos \varphi)\right]$$

where O represents the origin, $P(x,y)$ a variable point, $\underline{E}^{i}$ the elec-

([4]) G.TORALDO DI FRANCIA, Boll. Un. Mat. It., <u>11</u>, 332 (1956) tric field and $k = 2\pi/\text{wavelength}$. The electric field has unit intensity. The time factor $\exp(-i\omega t)$ will be understood.

It has been shown ([5]) that the induced current density $I(Q)$ (which, of course, has the direction of x) is a solution of the following integro-differential equation

$$(2) \qquad \left(\frac{\partial^2}{\partial x^2} + k^2\right) \iint_{\Sigma} I(Q)\, G(P,Q)\, d\Sigma_Q = \frac{i\,k}{Z}\, E^i_x(P)$$

where <u>both P and Q are points of Σ</u>, Z is the intrinsic impedance of empty space ([6]) and

$$(3) \qquad G(P,Q) = \frac{\exp(ik|P-Q|)}{4\pi|P-Q|}$$

is the free-space Green function.

As is customary for integral equations of the first kind ([7]), we will attempt to solve (2) by expanding $I(Q)$ as a series of a convenient set of functions. This will be done in two successive steps.

First, since the wavelength will be assumed to be large compared to the other linear dimensions involved (radius of the disc), it is natural to put

$$(4) \qquad I(Q) = I_o(Q) + ik\, I_1(Q) + (ik)^2\, I_2(Q) + \dots\dots$$

where $I_o(Q)$, $I_1(Q)$, $I_2(Q)$,.... are independent of k. We will assume that this expansion is possible and is uniformly convergent in the domain Σ for a non-vanishing interval $0 \leq k \leq k_1$.

For $z_P = 0$, we will put

$$(5) \qquad \frac{4\pi ik}{Z}\, E^i_x(P) = ik\, a_1 + (ik)^2 a_2\, y_P + (ik)^3\, a_3\, y_P^2 + \dots$$

where, by (1), the coefficients are found to be

([5]) G.TORALDO DI FRANCIA, Rend. Acc. Naz. Linc., <u>21</u>, 86 (1956).

([6]) The rationalized MKSA system will be employed throughout.

([7]) See for instance: P.M.MORSE and H.FESHBACH, <u>Methods of Theoretical Physics</u> (New York, 1953), Vol.I, p.925.

(6) $$a_n = \frac{4\pi}{Z} \frac{\sin^{n-1}\varphi}{(n-1)!}$$

By substituting (4) and (5) into (2), by expanding G(P,Q) as a power series of $ik|P-Q|$ and equating the coefficients of equal powers of ik on both sides, we find

(7) $$\frac{\partial^2}{\partial x^2} \iint_\Sigma \frac{I_m(Q)}{|P-Q|} d\Sigma_Q =$$

$$= a_m y_P^{m-1} + \sum_{o}^{m-2}{}_n \left\{ \frac{1}{(m-n-2)!} \iint_\Sigma I_n(Q) |P-Q|^{m-n-3} d\Sigma_Q - \right.$$

$$\left. - \frac{1}{(m-n)!} \frac{\partial^2}{\partial x_P^2} \iint_\Sigma I_n(Q) \ |P-Q|^{m-n-1} \ d\Sigma_Q \right\}$$

where $a_o = 0$ and the last sum is to be replaced by zero for $m < 2$.

Equation (7) represents a recurrent integro-differential equation for $I_m(Q)$, since the right side contains only the $I_n(Q)$'s with $n \leqslant m-2$, in addition to known quantities.

Solution of the Integro-Differential Equations for the First Terms of the Series Expansion.

In order to solve the successive equations which are obtained from (7), we will once more apply the procedure of expressing the solution as a sum of a convenient set of functions. Before doing this, it is useful to discuss the behavior of the current I(Q) at the rim of the screen Σ.

A boundary condition which seems to have general validity for an infinitely thin screen with ordinary (omnidirectional) conductivity is that the component of the current normal to the rim should vanish as $\sqrt{\rho}$, where ρ denotes the distance from the rim [8][9][10].

[8] A.W.MAUE, Z. Phys., 126, 601 (1949).

[9] D.S.JONES, Quart.Journ.Mech., 3, 420 (1950); 5, 363 (1952); Proc. Lond. Math. Soc., 2, 440 (1952).

[10] C.J.BOUWKAMP, Philips Res. Rep., 5, 401 (1950).

In our case, this seems to indicate that $I(Q)$ should vanish as $\sqrt{\rho}$ at all points of the rim where the tangent is not parallel to the x-axis. At the same conclusion one can arrive by considering the scattering by a small ellipsoid with unidirectional conductivity ([4]) and making the ellipsoid to degenerate into a thin disc.

Tentatively, we will assume that for any given value of k in the interval $0 \leq k \leq k_1$, the current $I(Q)$ may be expressed in the form $I(Q) = F(Q)\left[r^2-(Q-O)^2\right]^{1/2}$, where $F(Q)$ is a function which is regular in the domain Σ including the rim, and can be expanded into a power series of x_Q, y_Q. Precisely, we will assume that the coefficients $I_m(Q)$ of (4) may be expressed in the form

$$(8) \qquad I_m(Q) = \sum_{s+p+q\,=\,m-1} A_{spq}\, r^s\, x_Q^p\, y_Q^q \sqrt{r^2-(Q-O)^2}$$

where the coefficients have the dimensions of a surface density of current and s,p,q are positive integers or zero. Position (8) satisfies the requirement that $I_m(Q)$ be homogeneous of order m with respect to the lengths r, x_Q, y_Q, as is necessary on account of (4). However, it will be emphasized that no better justification of position (8) is offered than the ultimate success of the procedure.

Upon substitution of (8) into (7), we obtain after some rearrangements

$$(9) \qquad \sum_{s+p+q\,=\,m-1} A_{spq}\, r^s \frac{\partial^2}{\partial x_P^2} J_{pq0}(P) =$$

$$= a_m y_P^{m-1} + \sum_{u+v+w\,\leq\, m-3} A_{uvw}\, r^u \left[\frac{1}{(t-2)!} J_{vw(t-2)}(P) - \frac{1}{t!}\frac{\partial^2}{\partial x_P^2} J_{vwt}(P)\right]$$

where

$$(10) \qquad t = m - u - v - w - 1$$

and

$$(11) \qquad J_{pqt}(P) = \iint_{\Sigma} x_Q^p\, y_Q^q\, |P-Q|^{t-1} \sqrt{r^2-(Q-O)^2}\; d\Sigma_Q$$

The evaluation of the integrals $J_{pqt}(P)$ may be carried out as shown in the Appendix. It turns out that $J_{pqt}(P)$ is a polynomial of degree p+q+t+2 in r, x_P, y_P. Therefore, by substituting into (9) and equating the coefficients of equal powers, one obtains a recurrent system of linear equations for the coefficients A_{spq}. We do not want to enter into a general discussion about the solutions of these successive systems, which would be very involved. We will limit ourselves to saying that the solutions of the systems corresponding to the first few values of m have actually been determined and turn out to be unique.

Before presenting the results, we want to remind that our aim is the evaluation of the angular momentum absorbed by the screen. It will turn out that the terms in the expressions of $I_6(Q)$ and $I_8(Q)$ which are of odd degree in y_Q and the terms of the expression of $I_7(Q)$ which are of even degree in y_Q are not needed for our approximation. Accordingly, we have dispensed with their evaluation. Incidentally, we note that, on account of the symmetry of the problem, $I(Q)$ is an even function of x_Q.

We have obtained the following results:

(12) $$I_0(Q) = 0$$

(13) $$I_1(Q) = -\frac{2}{\pi^2} a_1 \sqrt{r^2-(Q-0)^2}$$

(14) $$I_2(Q) = -\frac{8}{3\pi^2} a_2 y_Q \sqrt{r^2-(Q-0)^2}$$

(15) $$I_3(Q) = \frac{1}{45\pi^2}\Big[(68 a_1+14 a_3) r^2 - (11 a_1+8 a_3) x_Q^2 - (59 a_1+152 a_3) y_Q^2\Big]\sqrt{r^2-(Q-0)^2}$$

(16) $$I_4(Q) = \left\{\frac{16}{9\pi^3} a_1 r^3 + \frac{4}{315\pi^2} y_Q \Big[(130 a_2+54 a_4) r^2 - (33 a_2+36 a_4) x_Q^2 - (129 a_2+324 a_4) y_Q^2\Big]\right\}\sqrt{r^2-(Q-0)^2}$$

(17) $$I_5(Q) = \frac{1}{135.105\,\pi^2}\Big[(834\,a_5 - 910\,a_3 - 12802\,a_1)\,r^4 +$$

$$+ (-528\,a_5 + 1435\,a_3 + 3679\,a_1)r^2x_Q^2 + (15312\,a_5 + 29323\,a_3 + 28255\,a_1)r^2y_Q^2 +$$

$$+ 12\,(12\,a_5 - 11\,a_3 - \frac{203}{16}\,a_1)\,x_Q^4 - 144(78\,a_5 + 61\,a_3 + \frac{841}{32}\,a_1)\,x_Q^2y_Q^2 -$$

$$- 12\,(5748\,a_5 + 2321\,a_3 + \frac{15083}{16}\,a_1)\,y_Q^4\Big]\sqrt{r^2 - (Q-0)^2}$$

(18) $$I_6(Q) = \frac{8}{27\pi^3}\Big[(\frac{6}{5}\,a_3 - \frac{196}{25}\,a_1)\,r^5 + a_1\,r^3x_Q^2 + \frac{29}{5}\,a_1\,r^3y_Q^2 +$$

$$+ \underline{\text{odd terms in } y_Q}\Big]\sqrt{r^2 - (Q-0)^2}$$

(19) $$I_7(Q) = \Big[-\frac{512}{2025\,\pi^3}\,r^5\,a_2y_Q + \underline{\text{even terms in } y_Q}\Big]\sqrt{r^2 - (Q-0)^2}$$

(20) $$I_8(Q) = \frac{1}{105\,\pi^3}\Big[(16\,a_5 - \frac{368}{9}\,a_3 + \frac{576544}{2835}\,a_1)\,r^7 +$$

$$+ (\frac{56}{9}\,a_3 - \frac{17128}{405}\,a_1)\,r^5x_Q^2 + \frac{8}{45}(203\,a_3 - \frac{15941}{9}\,a_1)\,r^5y_Q^2 +$$

$$+ \frac{2}{135}\,a_1r^3\,(97\,x_Q^4 + 2514\,x_Q^2y_Q^2 + 7537\,y_Q^4) + \underline{\text{odd terms in } y_Q}\Big]\sqrt{r^2 - (Q-0)^2}$$

Strictly speaking, we can only say that, if the series expansion exists and is uniformly convergent in a given interval $0 \leqslant k \leqslant k_1$, and the coefficients can be expressed in the form (8), then the first coefficients are given by the expressions (12)-(20). It remains an open question whether the series (4) converges at all. However, the first terms found should at least represent a good approximation to the exact solution, when k is sufficiently small. In other words, they should represent the first terms of an asymptotic expansion.

The lengthy and tedious calculations which lead to the formulas (12)-(20) have been repeated several times by different persons, and one may be fairly confident that the results given are correct.

Angular Momentum Absorbed by the Screen.

Let us now consider the whole wave with elliptic polarization. To do this, we have to introduce, beside the wave (1), a second wave whose electric field is in the direction $\underline{s} \wedge \underline{i}$, i.e., perpendicular to both the direction of propagation and the direction of conductivity. Precisely, the total field will be now expressed by

$$E^i(P) = (\underline{i} + ie\, \underline{s} \wedge \underline{i}) \exp\left[ik\, \underline{s}.(P-O)\right] \tag{21}$$

where e is a complex quantity which determines the shape of the elliptic polarization. By applying some standard formulas ([11]), it can be proved that the polarization is linear if Re $e = 0$ and is circular if $e = \pm 1$. It can also be shown that, if the polarization differs only very little from circular polarization, so that we can put $e = \pm(1+\varepsilon)$ with $|\varepsilon|$ very small, the ratio $a/b = 1+\alpha$ of the axes of the ellipse differs from unity by a very small amount α and, apart from higher order infinitesimals, $|\alpha| = |\varepsilon|$. We shall call α the <u>ellipticity</u> of the polarization.

If we define the <u>complex scattering cross section</u> σ of Σ by

$$\sigma = \frac{\Sigma}{1+ee^*} \iint_\Sigma I^*(Q) \exp\left[ik\, \underline{s}.(Q-O)\right] d\Sigma_Q \tag{22}$$

it can be shown ([12]) that the ordinary scattering cross section $\bar{\sigma}$ of Σ for the wave (21) is equal to Re σ and that the average torque M about $\underline{s}$ experienced by the screen is expressed by

$$M = \frac{S}{\omega} \mathrm{Re}\,(e\sigma) \tag{23}$$

where S represents the average Poynting vector of the wave (21).

It is seen from (23) that when the polarization is circular, $(e = \pm 1)$, there results

$$|M| = \frac{S}{\omega} \bar{\sigma} \tag{24}$$

Since the non-orbital angular momentum carried by the wave per unit time per unit surface is S/ω, we see that in this case <u>the angular momentum cross section equals the ordinary scattering cross section</u>.

However, if the polarization is not circular, it is seen by (23) that, generally speaking, also the imaginary part of σ ap-

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([11]) See, for instance: G.TORALDO DI FRANCIA, <u>Electromagnetic Waves</u>, (New York, 1956), p.152

([12]) See Ref. (4).

pears in the expression of M . If M_o denotes the torque corresponding to circular polarization and M_α the torque corresponding to a small ellipticity α for one and the same value of S, the quantity $(M_\alpha - M_o)/M_o$ wil be termed the relative error. It is readily found by (23) that the maximum absolute value of the relative error is given by

$$(25) \qquad \left| \frac{M_\alpha - M_o}{M_o} \right|_{max} = |\alpha| \sqrt{1 + \left(\frac{\mathrm{Im}\ \sigma}{\mathrm{Re}\ \sigma}\right)^2}$$

Therefore, in order to determine the error which may be introduced in the measurement of the spin by an inaccurate production of circular polarization, we have also to know the imaginary part of the complex scattering cross section σ. The error can be quite substantial, even for a very small $|\alpha|$, when $\mathrm{Im}\ \sigma$ is large compared to $\mathrm{Re}\ \sigma$. We shall see that this actually occurs when r is very small compared to the wavelength.

<u>The First Terms of the Expansion of the Complex Scattering Cross Section.</u>

By substituting (12)-(20) into (22), by expanding the exponential of (22) in a power series of $ik\,\underline{s}\cdot(Q-O) = ik\, y_Q \sin\varphi$ and remembering (6), it is found that the first three terms of the expression of the ordinary scattering cross section $\bar{\sigma}$ are given by

$$(26) \qquad \bar{\sigma} = \mathrm{Re}\ \sigma = \frac{2}{1+ee^*}\ \frac{64}{27\pi}\, k^4 r^6 \left[1 + \left(\frac{27}{25} - \frac{1}{5}\sin^2\varphi\right)(kr)^2 + \right.$$

$$\left. + \left(\frac{4682}{6125} - \frac{1144}{7875}\sin^2\varphi + \frac{3}{175}\sin^4\varphi\right)(kr)^4 + \ldots \right]$$

while the first three terms of the imaginary cross section are given by

$$(27) \qquad \mathrm{Im}\ \sigma = \frac{2}{1+ee^*}\ \frac{8}{3}\, k\, r^3 \left[1 + \frac{1}{5}\left(3 + \frac{1}{3}\sin^2\varphi\right)(kr)^2 + \right.$$

$$\left. + \frac{1}{525}\left(139 - \frac{4}{3}\sin^2\varphi - \sin^4\varphi\right)(kr)^4 + \ldots \right]$$

In the particular case of normal incidence ($\varphi=0$) and linear polarization parallel to x ($e=0$), we have

$$(28) \qquad \bar{\sigma} = \frac{128}{27\pi} k^4 r^6 \left[1 + \frac{27}{25}(kr)^2 + \frac{4682}{6125}(kr)^4 + \dots\right]$$

$$(29) \quad \text{Im}\,\sigma = -\frac{16}{3} k\, r^3 \left[1 + \frac{3}{5}(kr)^2 + \frac{139}{525}(kr)^4 + \dots\right]$$

It is interesting to compare (28) with the expression

$$(30) \qquad \bar{\sigma} = \frac{16}{27\pi} k^4 r^6 \left[1 + \frac{8}{25}(kr)^2 + \frac{311}{6125}(kr)^4 + \dots\right]$$

which is valid for the scattering at normal incidence of an acoustic wave by a rigid disc [13][14], and with the expression

$$(31) \qquad \bar{\sigma} = \frac{128}{27\pi} k^4 r^6 \left[1 + \frac{22}{25}(kr)^2 + \frac{7312}{18375}(kr)^4 + \dots\right]$$

which is valid for the scattering at normal incidence of an elctromagnetic wave by a disc with ordinary conductivity [15]. This comparison seems to suggest that when kr is small, the acoustic case is the most favourable for the representation of $\bar{\sigma}$ by the first few terms of the expansion, while the case treated in the present paper is the least favourable.

The three cases are shown in Fig.1 where the ratio of the scattering cross section to the geometric cross section $\sigma_g = \pi r^2$ is plotted against kr.

In the case of grazing incidence ($\varphi=\pi/2$) and linear polarization parallel to x ($e=0$), the cross sections become

$$(32) \qquad \bar{\sigma} = \frac{128}{27\pi} k^4 r^6 \left[1 + \frac{22}{25}(kr)^2 + \frac{1403}{2205}(kr)^4 + \dots\right]$$

$$(33) \quad \text{Im}\,\sigma = \frac{16}{3} k\, r^3 \left[1 + \frac{2}{3}(kr)^2 + \frac{82}{315}(kr)^4 + \dots\right]$$

Upon comparison of (32) and (28), we see that in the case of grazing

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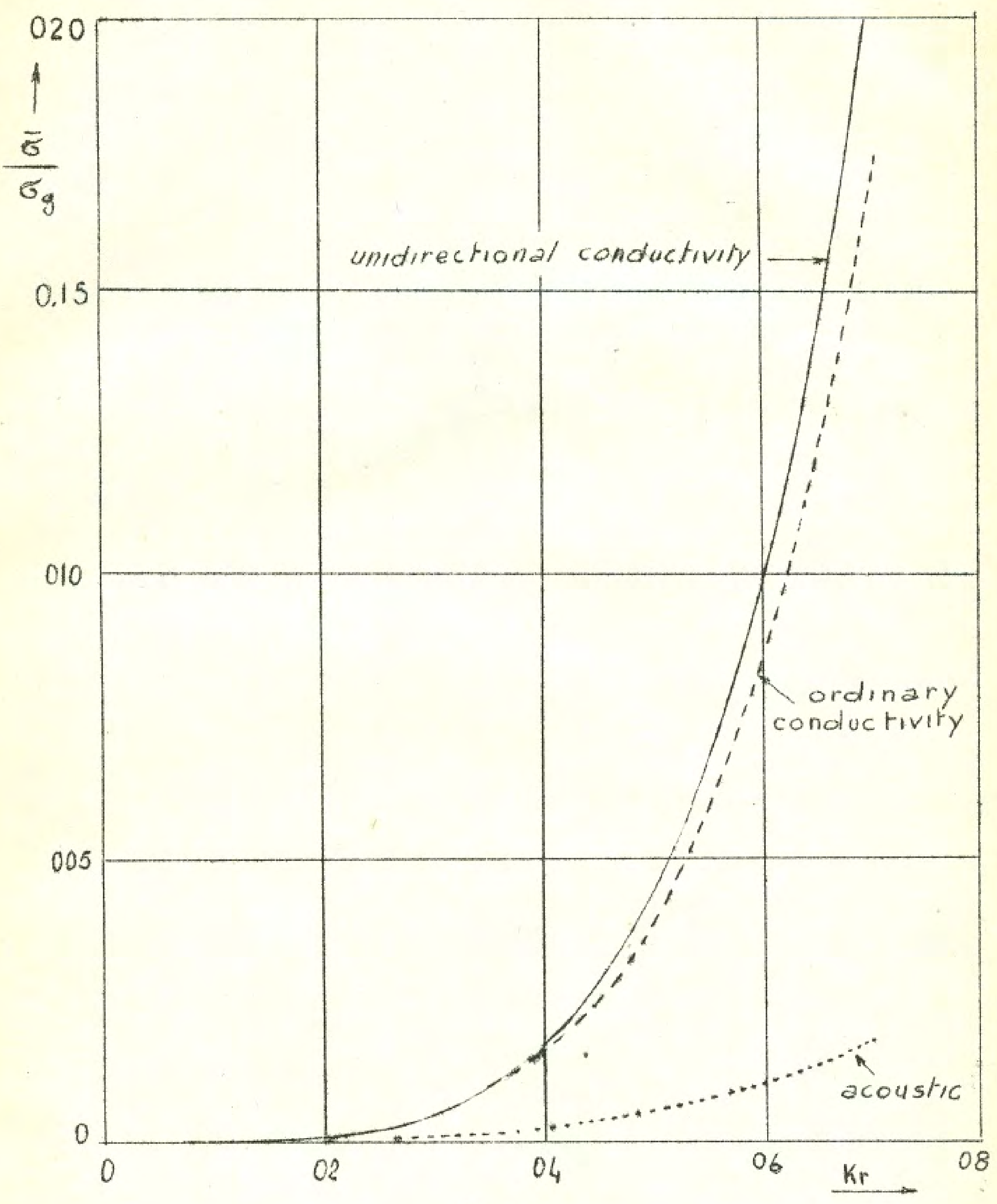
([13]) C.J.BOUWKAMP, Phys. Rev., 75, 1608 (1949); Physica, 26, 1 (1950).
([14]) H.LEVINE and J.SCHWINGER, Phys. Rev., 74, 958 (1948).
([15]) C.J.BOUWKAMP, Rep. Progr. Phys., 17, 35 (1954).



020
0.15
010
005
0
$\frac{\bar{\sigma}}{\sigma_g}$
unidirectional conductivity
ordinary conductivity
acoustic
0
02
04
06
08
Kr

Fig. 1

incidence the scattering cross section increases less rapidly than in the case of normal incidence.

Whatever the angle of incidence, when kr is small, it is found $\mathrm{Im}\,\sigma/\bar{\sigma} = 9\pi/(2kr)^3$, apart from terms of smaller order. By substituting into (25), we can fix an upper limit for the values of $|\alpha|$ which are allowed if we do not want to exceed a certain maximum value for the relative error. It is found

$$(34) \qquad |\alpha| < \frac{(2kr)^3}{9\pi} \left|\frac{M_\alpha - M_o}{M_o}\right|_{max}$$

For example, suppose that we want to make a measurement of the spin with $kr = 1/2$ and with 1% accuracy. There follows from (34) that the ellipticity should not exceed .04% . This would be a very exacting requirement.

Evaluation of the Scattered Field on the Screen.

It is of interest to evaluate the field generated by the induced current, namely the scattered field, on the disc Σ. Especially the components E_y and H_z , which in our case are not necessarily equal and opposite to those of the incident field, are likely to give us some insight as to the different behavior of a screen with unidirectional conductivity, compared with the behavior of a screen with omnidirectional conductivity.

The incident wave will be assumed to have the form (1). By constructing the vector potential generated by the current $I(Q)$ and applying standard formulas, we find for the components E_x and E_y of the scattered field

$$(35) \qquad E_x(P) = -\frac{Z}{ik}\left(k^2 + \frac{\partial^2}{\partial x_P^2}\right).\iint_\Sigma I(Q)\, G(P,Q)\, d\Sigma_Q$$

$$(36) \qquad E_y(P) = -\frac{Z}{ik}\,\frac{\partial^2}{\partial x_P \partial y_P}\iint_\Sigma I(Q)\, G(P,Q)\, d\Sigma_Q$$

If $I(Q)$ represents the exact solution of the integro-differential equation (2), we obviously get from (35) $E_x(P) = -E_x^i(P)$.

Now, equation (35) can be interpreted as a differential equation for the integral on the right side. Since $I(Q)$ is an even function of x_Q, it is readily proved that the integral is an even function of x_P. On the other hand, $E_x^i(P)$ depends solely on y_P. Therefore, we get upon integration of the differential equation

$$(37) \qquad \frac{Z}{ik} \iint_\Sigma I(Q)\, G(P,Q)\, d\Sigma_Q = \frac{1}{k^2} E_x^i(P) + F(y_P) \cos(ky_P)$$

where $F(y_P)$ is an unknown function. By substituting (12)-(20) into (37), by expanding $G(P,Q)$ and $E_x^i(P)$ in power series, and utilizing the integrals $J_{pqt}(P)$ evaluated in the Appendix, one can derive the expression of $F(y_P)$. Its first derivative turns out to be ([16])

$$(38) \qquad F'(y) = \frac{Z}{4\pi} \left\{ \frac{a_2}{ik} + y\,(a_1 + 2\,a_3) + ik \left[-\frac{2}{3} a_2 r^2 + \frac{3}{2} y^2 (a_2 + 2\,a_4) \right] + \right.$$
$$+ (ik)^2 \left[-\frac{1}{5} r^2 y\,(7\,a_1 + 6\,a_3) + \frac{1}{6} y^3 (5\,a_1 + 12\,a_3 + 24\,a_5) \right] +$$
$$+ (ik)^3 \left[\frac{1}{315} r^4 (130\,a_2 - 9\,a_4) - \frac{4}{3\pi} r^3 a_1 y - \right.$$
$$\left. \left. - \frac{1}{7} r^2 y^2 (11\,a_2 + 12\,a_4) + \frac{5}{24} y^4 (5\,a_2 + 12\,a_4 + 24\,a_6) \right] + \dots \right\}$$

apart from terms of higher order in k.

Now, by substituting (37) into (36), we get

$$(39) \qquad E_y = k\, F'(y) \sin(kx)$$

By combining (39) with (38), we get the expression of E_y exact with respect to x and approximate up to the term in k^4 with respect to y. In the case of normal incidence ($\varphi=0$) we obtain by recalling (6)

$$(40) \qquad E_y = ky \left[1 + (kr)^2 \left(\frac{7}{5} - \frac{5}{6} \frac{y^2}{r^2} \right) + i\, \frac{4}{3\pi} (kr)^3 + \dots \right] \sin(kx)$$

and in the case of grazing incidence ($\varphi=\pi/2$)

([16]) When no confusion is possible we simply write x, y for x_P, y_P.

(41) $$E_y = -i\left[1 + 2iky + 2(kr)^2\left(\frac{1}{3} - \frac{y^2}{r^2}\right) + 2i(kr)^3\frac{y}{r}\left(1 - \frac{y^2}{r^2}\right) + (kr)^4\left(\frac{257}{630} - \frac{4}{3\pi}\frac{y}{r} - \frac{13}{7}\frac{y^2}{r^2} + \frac{3}{2}\frac{y^4}{r^4}\right) + \ldots\right]\sin(kx)$$

It will be noted that E_y is of the order of $(kr)^2$ in the former case and of the order of kr in the latter case.

From the knowledge of I(Q) and the equation of continuity we can derive the value of the surface density of charge on Σ, then by a standard relation the value of E_z, which is equal and opposite on both sides of Σ. Precisely, we get

(42) $$E_z = \pm\frac{Z}{2ik}\frac{\partial I}{\partial x}$$

Upon substitution, we obtain in the case of normal incidence

(43) $$E_z = \pm\frac{4x}{\pi\sqrt{r^2-x^2-y^2}}\left[1 + \frac{1}{30}(kr)^2\left(30 - 11\frac{x^2}{r^2} - 27\frac{y^2}{r^2}\right) + i\frac{8}{9\pi}(kr)^3 + \ldots\right]$$

and in the case of grazing incidence

(44) $$E_z = \pm\frac{4x}{\pi\sqrt{r^2-x^2-y^2}}\left[1 + \frac{4}{3}iky + \frac{1}{6}(kr)^2\left(7 - 3\frac{x^2}{r^2} - 11\frac{y^2}{r^2}\right) + i(kr)^3\left(\frac{8}{9\pi} + \frac{62}{45}\frac{y}{r} - \frac{26}{35}\frac{x^2y}{r^3} - \frac{58}{35}\frac{y^3}{r^3}\right) + \ldots\right]$$

In both cases E_z becomes infinite at the rim of Σ.

As to the magnetic field, $H_x = 0$, because the current is in the direction of the x-axis, while $H_y = \mp I(Q)/2$ depending on which side of Σ is considered. Thus for normal incidence it is found

(45) $$H_y = \pm\frac{4ik}{\pi Z}\sqrt{r^2-x^2-y^2}\left[1 + \frac{1}{90}(kr)^2\left(68 - 11\frac{x^2}{r^2} - 59\frac{y^2}{r^2}\right) + \frac{8}{9\pi}i(kr)^3 + \ldots\right]$$

and for grazing incidence

(46) $$H_y = \pm\frac{4ik}{\pi Z}\sqrt{r^2-x^2-y^2}\left[1 + \frac{4}{3}iky + \frac{1}{90}(kr)^2\left(75 - 15\frac{x^2}{r^2} - 135\frac{y^2}{r^2}\right) + i(kr)^3\left(\frac{8}{9\pi} + \frac{278}{315}\frac{y}{r} - \frac{26}{105}\frac{yx^2}{r^3} - \frac{122}{105}\frac{y^3}{r^3}\right) + \ldots\right]$$

In all cases, H_y vanishes at the rim of Σ.

Finally, the value of H_z can be derived from (40), (41) by applying the second Maxwell equation. For normal incidence it is found

$$(47) \qquad H_z = -\frac{ik}{Z} y \left[1 + (kr)^2\left(\frac{7}{5} - \frac{5}{6}\frac{y^2}{r^2}\right) + i\,\frac{4}{3\pi}(kr)^3 + \ldots\right]\cos(kx)$$

and for grazing incidence

$$(48) \qquad H_z = -\frac{1}{Z}\left[1 + 2iky + 2(kr)^2\left(\frac{1}{3} - \frac{y^2}{r^2}\right) + 2i\,(kr)^3\,\frac{y}{r}\left(1 - \frac{y^2}{r^2}\right) + \right.$$
$$\left. + (kr)^4\left(\frac{257}{630} - \frac{4}{3\pi}\frac{y}{r} - \frac{13}{7}\frac{y^2}{r^2} + \frac{3}{2}\frac{y^4}{r^4}\right) + \ldots\right]\cos(kx)$$

Thus, we have completed the evaluation of the scattered field over the screen Σ.

An interesting result which we want to stress is that <u>the components E_y, H_z of the scattered field are, on Σ, periodic with respect to x with the periodicity equal to the wavelength</u>. It is rather puzzling to note that this result is readily proved to be valid for any shape of Σ (provided its rim is met not more than twice by any line parallel to the x-axis, and the incident field does not depend on x). In the particular case when Σ is circular (or, more generally, symmetric with respect to the y axis), the straight lines $x = 0$, $x = \pm\lambda/2$, $x = \pm 2\lambda/2$ etc. are nodal lines for E_y and anti-nodal lines for H_z.

Conclusion.

As a result of our analysis, we may conclude that an accurate measurement of the spin of a circularly polarized wave by means of a small screen Σ is not an easy matter. If kr is very small, the real cross section becomes negligible compared to the imaginary cross section. We have seen that in this case <u>even an extremely small degree of ellipticity of the polarization may bring about a considerable error in the measurement</u>. On the other hand, if kr approaches or exceeds unity, the theory developed breaks down, because the series (26) converges only very slowly (if at all). At present, the best way to proceed should consist in making kr very small and repeating the measurement at several different azimuths, so that the effect of ellipticity averages to zero.

The experiment may be carried out by suspending the disc and measuring the deflection produced by the torque. The direction of propagation of the wave is along the vertical, while the plane of the disc may be either horizontal or vertical. Comparison of (26) and (32) seems to suggest that our approximation is a little better in the case of the vertical disc.

In any case, it is desirable that the problem be tackled also from the direction opposite to that of this paper, namely starting from large values of kr and setting up an asymptotic expansion.

Some byproducts of our investigation are worth mentioning. For instance, it turns out that, at least for a given range of values of kr , the scattering cross section is larger for unidirectional conductivity than for omnidirectional conductivity (see Fig.1). This suggests the intriguing problem of determining what electromagnetic properties should be given to Σ for obtaining maximum cross section. Another interesting result is the exact form of the dependence on x for the components E_y, H_z of the scattered field over Σ, expressed by the formulas (40), (41), (47), (48). These components characterize the difference from the case of omnidirectional conductivity.

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Appendix.

The integrals $J_{pqt}(P)$ of equation (11) may be evaluated as follows.

First we distinguish two cases, according as t is odd or even. If t is odd and we put $t = 2h+1$, we get

$$(49)\quad J_{pqt}(P) = \iint_{\Sigma} x_Q^p\, y_Q^q\, (P-Q)^{2h} \sqrt{r^2-(Q-0)^2}\; d\Sigma_Q =$$

$$= \iint_{\Sigma} x_Q^p\, y_Q^q \left[(P-0)^2+ (Q-0)^2- 2(P-0).(Q-0)\right]^h \sqrt{r^2-(Q-0)^2}\; d\Sigma_Q$$

By introducing polar coordinates with $x_Q = \rho \cos \Theta$, $y_Q = \rho \sin \Theta$, we have

$$(50)\quad J_{pqt}(P) = \int_0^{2\pi} \cos^p\Theta \sin^q\Theta\, d\Theta \int_0^r \rho^{p+q+1} \left[x_P^2+y_P^2+\rho^2 -2\rho(x_P\cos \Theta +y_P\sin \Theta\right]^h \cdot \sqrt{r^2- \rho^2}\; d\rho$$

The integration is straightforward for every value of h. We list some of the results for tnis case, which have been applied for the solution of the integro-differential equations (7):

$$J_{001} = \frac{2}{3} \pi r^3$$

$$J_{021} = \frac{2}{15} \pi r^5 \quad , \quad J_{003} = \frac{2}{15} \pi r^3 \left[5(x^2+ y^2) + 2 r^2\right]$$

$$J_{013} = - \frac{4}{15} \pi r^5 y$$

$$J_{041} = \frac{2}{35} \pi r^7 \quad , \quad J_{221} = \frac{2}{105} \pi r^7$$

$$J_{203} = \frac{2}{15} \pi r^5 \left\{x^2+ y^2+ \frac{4}{7} r^2\right]$$

$$J_{005} = \frac{2}{3} \pi r^3 \left[(x^2+ y^2)^2 + \frac{8}{5} r^2(x^2+ y^2) + \frac{8}{35} r^4\right]$$

It turns out in this case that $J_{qpt} = J_{pqt}$.

When t is even, the integration is a little less straightforward. It is expedient to introduce for a moment polar coordinates $|Q-P|$, φ, with the origin at P [17] and to put $d\Sigma = |Q-P|\, d\varphi\, |dQ|$ (Fig.2)

[17] H.A.BETHE, Phys. Rev., 66, 163 (1944).

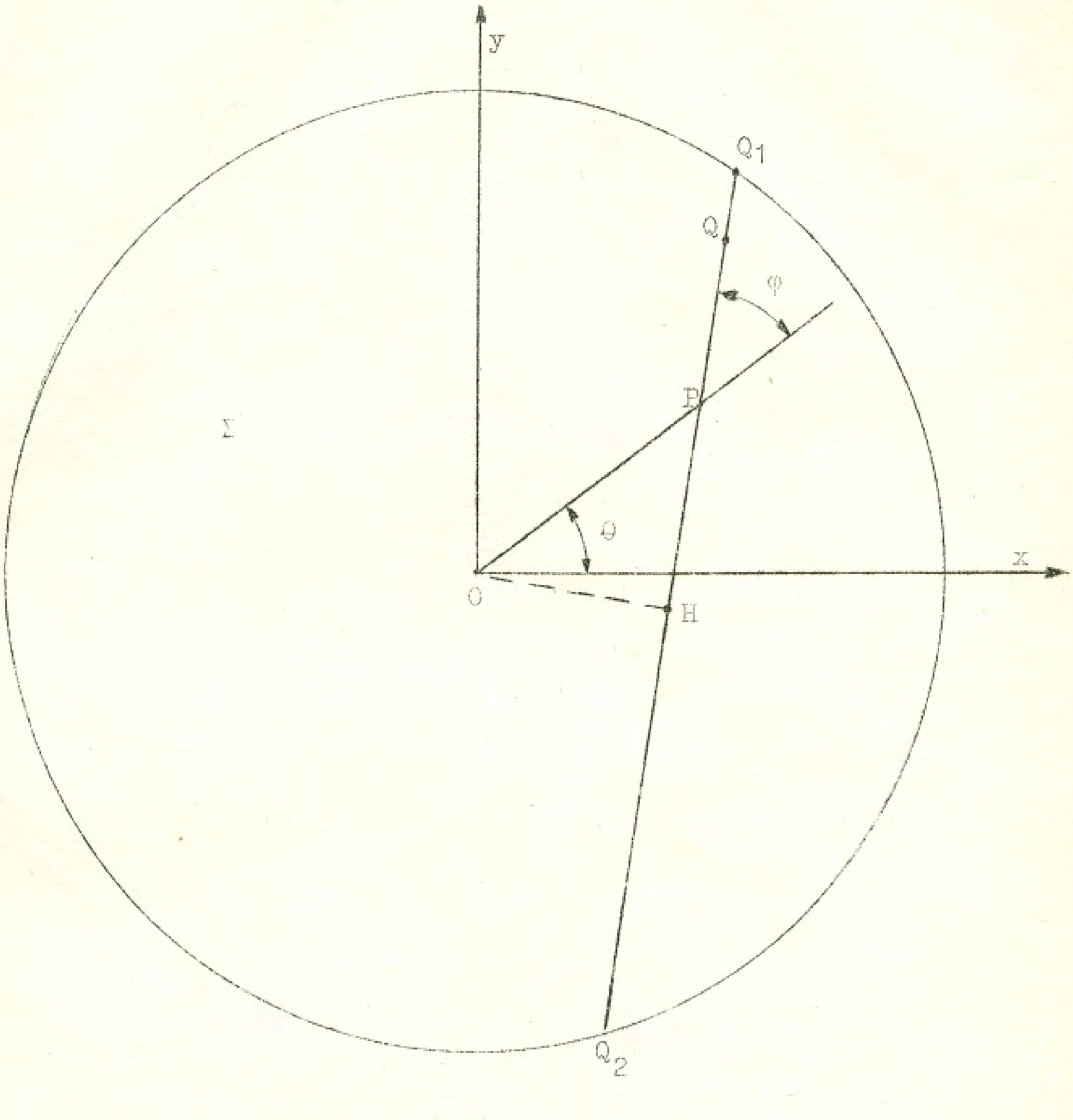

Fig. 2

with the understanding that dQ has the same direction as Q-P. By putting $t = 2h$, we may write

$$(51) \qquad J_{pqt}(P) = \int_0^{2\pi} d\varphi \int_P^{Q_1} x_Q^p \, y_Q^q \, (P-Q)^{2h} \sqrt{r^2 - (Q-O)^2} \, |dQ|$$

We may add to the integral with respect to Q its value for the angle $\varphi+\pi$, obtaining in this way the integral evaluated from Q_2 to Q_1. Accordingly, φ will vary only from 0 to π. Further we note that

$$(52) \qquad r^2 - (Q-O)^2 = (Q_1-O)^2 - (H-O)^2 - (Q-H)^2 = (Q_1-H)^2 - (Q-H)^2$$

Thus, we obtain

$$(53) \qquad J_{pqt}(P) = \int_0^{2\pi} d\varphi \int_{Q_2}^{Q_1} x_Q^p \, y_Q^q \, (P-Q)^{2h} \sqrt{(Q_1-H)^2 - (Q-H)^2} \, |dQ|$$

Next, if we put $|P-O| = \rho$ and denote by u the distance HQ, counted positive upwards and negative downwards, we can write

$$(54) \qquad (P-Q)^2 = (P-H)^2 + (Q-H)^2 - 2\,(P-H).(Q-H) = \rho^2 \cos^2\varphi + u^2 - 2\rho u \cos\varphi$$

and

$$(55) \qquad x_Q = u \cos(\Theta+\varphi) + \rho \sin\varphi \sin(\Theta+\varphi)$$

$$(56) \qquad y_Q = u \sin(\Theta+\varphi) - \rho \sin\varphi \cos(\Theta+\varphi)$$

Finally, since $(Q_1-H)^2 = r^2 - \rho^2 \sin^2\varphi$, the integral (53) becomes

$$(57) \qquad J_{pqt}(P) = \int_0^{\pi} d\varphi \int_{-\sqrt{r^2-\rho^2\sin^2\varphi}}^{+\sqrt{r^2-\rho^2\sin^2\varphi}} \left[u \cos(\Theta+\varphi) + \rho \sin\varphi \sin(\Theta+\varphi)\right]^p \times$$

$$\times \left[u \sin(\Theta+\varphi) - \rho \sin\varphi \cos(\Theta+\varphi)\right]^q \left[\rho^2 \cos^2\varphi + u^2 - 2u\rho \cos\varphi\right]^h \times$$

$$\times \sqrt{r^2 - \rho^2 \sin^2\varphi - u^2} \; du$$

Upon expansion, this transforms into a sum of elementary integrals. Some of the results obtained, which have been applied for the solution of the integro-differential equations (7) are listed below:

$$J_{000} = \frac{\pi^2}{2}\left[r^2 - \frac{1}{2}(x^2 + y^2)\right]$$

$$J_{010} = \frac{\pi^2}{4}\, y \left[r^2 - \frac{3}{4}(x^2 + y^2)\right]$$

$$J_{020} = \frac{\pi^2}{16}\left[r^4 + \frac{1}{2}\, r^2(5y^2 - x^2) - \frac{1}{8}(19\, y^4 + 18\, x^2y^2 - x^4)\right]$$

$$J_{002} = \frac{\pi^2}{8}\left[r^4 + r^2(x^2 + y^2) - \frac{1}{8}(x^2+y^2)^2\right]$$

$$J_{030} = \frac{\pi^2}{64}\, y \left[3r^4 + r^2(7y^2 - 3x^2) + \frac{5}{16}\,(x^2 + y^2)(3\, x^2 - 25\, y^2)\right]$$

$$J_{210} = \frac{\pi^2}{64}\, y \left[r^4 - r^2(y^2 - 9\, x^2) + \frac{5}{16}(x^2+y^2)(y^2 - 27\, x^2)\right]$$

$$J_{012} = -\frac{\pi^2}{16}\, y\left[r^4 - \frac{1}{2}\, r^2(x^2 + y^2) + \frac{1}{8}(x^2+y^2)^2\right]$$

$$J_{040} = \frac{\pi^2}{128}\left[3r^6 - \frac{3}{2}\, r^4(x^2 - 3y^2) + \frac{1}{16}\, r^2(9x^4 - 102\, x^2y^2 + 169\, y^4) - \right.$$

$$\left. - \frac{3}{32}\,(x^2 + y^2)(x^4 - 26\, x^2y^2 + 141\, y^4)\right]$$

$$J_{220} = \frac{\pi^2}{128}\left[r^6 + \frac{1}{2}\, r^4(x^2 + y^2) - \frac{1}{16}\, r^2(17\, x^4 - 246\, x^2y^2 + 17\, y^4) + \right.$$

$$\left. + \frac{1}{32}\,(x^2+y^2)(13\, x^4 - 478\, x^2y^2 + 13\, y^4)\right]$$

$$J_{022} = \frac{\pi^2}{32}\left[r^6 + \frac{1}{4}\, r^4(3x^2+y^2) - \frac{1}{8}\, r^2(x^2+y^2)(x^2-3y^2) + \frac{1}{64}(x^2+y^2)^2(x^2-9y^2)\right]$$

$$J_{004} = \frac{\pi^2}{16}\left[r^6 + \frac{9}{2}\, r^4(x^2+y^2) + \frac{9}{8}\, r^2(x^2 + y^2)^2 - \frac{1}{16}\,(x^2+y^2)^3\right]$$

$$J_{032} = -\frac{\pi^2}{128}\, y \left[3r^6 - \frac{1}{2}\, r^4(3x^2+y^2) + \frac{1}{16}\, r^2(x^2+y^2)(9x^2 - 11\, y^2) - \right.$$

$$\left. - \frac{1}{32}\,(x^2+y^2)^2(3x^2 - 11\, y^2)\right]$$

$$J_{212} = -\frac{\pi^2}{128}\, y\left[r^6 + \frac{1}{2}\, r^4(x^2-y^2) - \frac{1}{16}\, r^2(x^2+y^2)(17x^2 - 3y^2) + \right.$$

$$\left. + \frac{1}{32}\,(x^2+y^2)^2(13\, x^2 - y^2)\right]$$

$$J_{014} = - \frac{3\pi^2}{32} y \left[r^6 + \frac{3}{4} r^4 (x^2+y^2) - \frac{1}{8} r^2 (x^2+y^2)^2 + \frac{1}{64} (x^2+y^2)^3 \right]$$

It is to be noted that when t is even, J_{qpt} may be obtained from J_{pqt} by interchanging x and y. Accordingly, we have dispensed with giving the expression of J_{qpt} when J_{pqt} was already given.